

X100/701

NATIONAL
QUALIFICATIONS
2005

FRIDAY, 20 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**



Answer all the questions.

1. (a) Given $f(x) = x^3 \tan 2x$, where $0 < x < \frac{\pi}{4}$, obtain $f'(x)$. 3
- (b) For $y = \frac{1+x^2}{1+x}$, where $x \neq -1$, determine $\frac{dy}{dx}$ in simplified form. 3
2. Given the equation $2y^2 - 2xy - 4y + x^2 = 0$ of a curve, obtain the x -coordinate of each point at which the curve has a horizontal tangent. 4
3. Write down the Maclaurin expansion of e^x as far as the term in x^4 . 2
 Deduce the Maclaurin expansion of e^{x^2} as far as the term in x^4 . 1
 Hence, or otherwise, find the Maclaurin expansion of e^{x+x^2} as far as the term in x^4 . 3
4. The sum, $S(n)$, of the first n terms of a sequence, u_1, u_2, u_3, \dots is given by $S(n) = 8n - n^2$, $n \geq 1$.
 Calculate the values of u_1, u_2, u_3 and state what type of sequence it is. 3
 Obtain a formula for u_n in terms of n , simplifying your answer. 2
5. Use the substitution $u = 1 + x$ to evaluate $\int_0^3 \frac{x}{\sqrt{1+x}} dx$. 5
6. Use Gaussian elimination to solve the system of equations below when $\lambda \neq 2$:
- $$\begin{array}{rrcr} x & + & y & + & 2z & = & 1 \\ 2x & + & \lambda y & + & z & = & 0 \\ 3x & + & 3y & + & 9z & = & 5. \end{array}$$
- 4
- Explain what happens when $\lambda = 2$. 2
7. Given the matrix $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$, show that $A^2 + A = kI$ for some constant k , where I is the 3×3 unit matrix. 4
 Obtain the values of p and q for which $A^{-1} = pA + qI$. 2
8. The equations of two planes are $x - 4y + 2z = 1$ and $x - y - z = -5$. By letting $z = t$, or otherwise, obtain parametric equations for the line of intersection of the planes. 4
 Show that this line lies in the plane with equation

$$x + 2y - 4z = -11.$$
 1

9. Given the equation $z + 2i\bar{z} = 8 + 7i$, express z in the form $a + ib$.

10. Prove by induction that, for all positive integers n ,

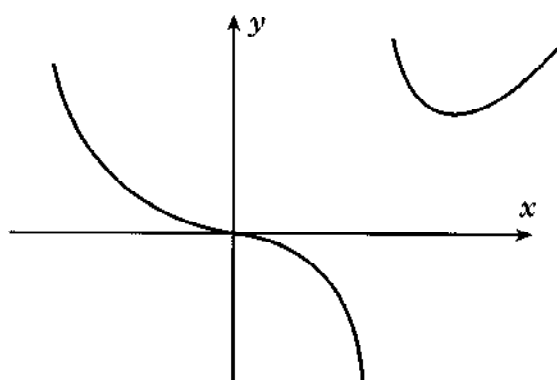
$$\sum_{r=1}^n \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

5

State the value of $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{r(r+1)(r+2)}$.

1

11. The diagram shows part of the graph of $y = \frac{x^3}{x-2}$, $x \neq 2$.



(a) Write down the equation of the vertical asymptote.

1

(b) Find the coordinates of the stationary points of the graph of $y = \frac{x^3}{x-2}$.

4

(c) Write down the coordinates of the stationary points of the graph of $y = \left| \frac{x^3}{x-2} \right| + 1$.

2

12. Let $z = \cos \theta + i \sin \theta$.

(a) Use the binomial expansion to express z^4 in the form $u + iv$, where u and v are expressions involving $\sin \theta$ and $\cos \theta$.

3

(b) Use de Moivre's theorem to write down a second expression for z^4 .

1

(c) Using the results of (a) and (b), show that

$$\frac{\cos 4\theta}{\cos^2 \theta} = p \cos^2 \theta + q \sec^2 \theta + r, \text{ where } -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

stating the values of p , q and r .

6

[Turn over for Questions 13, 14 and 15 on Page four]

13. Express $\frac{1}{x^3 + x}$ in partial fractions.

4

Obtain a formula for $I(k)$, where $I(k) = \int_1^k \frac{1}{x^3 + x} dx$, expressing it in the form $\frac{a}{b}$, where a and b depend on k .

4

Write down an expression for $e^{I(k)}$ and obtain the value of $\lim_{k \rightarrow \infty} e^{I(k)}$.

2

14. Obtain the general solution of the differential equation

$$\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = 20 \sin x.$$

7

Hence find the particular solution for which $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.

3

15. (a) Given $f(x) = \sqrt{\sin x}$, where $0 < x < \pi$, obtain $f'(x)$.

1

- (b) If, in general, $f(x) = \sqrt{g(x)}$, where $g(x) > 0$, show that $f'(x) = \frac{g'(x)}{k\sqrt{g(x)}}$, stating the value of k .

2

Hence, or otherwise, find $\int \frac{x}{\sqrt{1-x^2}} dx$.

3

- (c) Use integration by parts and the result of (b) to evaluate

$$\int_0^{1/2} \sin^{-1} x \, dx.$$

4

[END OF QUESTION PAPER]