X100/701

NATIONAL QUALIFICATIONS 2003

WEDNESDAY, 21 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.
2. There are five Sections in this paper.

   Section A assesses the compulsory units Mathematics 1 and 2
   Section B assesses the optional unit Mathematics 3
   Section C assesses the optional unit Statistics 1
   Section D assesses the optional unit Numerical Analysis 1
   Section E assesses the optional unit Mechanics 1.

   Candidates must attempt Section A (Mathematics 1 and 2) **and one** of the following Sections:

   Section B (Mathematics 3)
   Section C (Statistics 1)
   Section D (Numerical Analysis 1)
   Section E (Mechanics 1).
3. **Candidates must use a separate answer book for each Section.** Take care to show clearly the optional section chosen. On the front of the answer book, in the top right hand corner, write B, C, D or E.
4. A booklet of Mathematical Formulae and Statistical Tables is supplied for all candidates. It contains Numerical Analysis formulae and Statistical formulae and tables.
5. **Full credit will be given only where the solution contains appropriate working.**

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Section A (Mathematics 1 and 2)

All candidates should attempt this Section.

Answer all the questions.

Marks

A1. (*a*) Given $f(x) = x(1 + x)^{10}$, obtain $f'(x)$ and simplify your answer. **3**

(*b*) Given $y = 3^x$, use logarithmic differentiation to obtain $\frac{dy}{dx}$ in terms of x. **3**

A2. Given that $u_k = 11 - 2k$, $(k \geq 1)$, obtain a formula for $S_n = \sum_{k=1}^{n} u_k$. **3**
Find the values of n for which $S_n = 21$. **2**

A3. The equation $y^3 + 3xy = 3x^2 - 5$ defines a curve passing through the point A (2, 1). Obtain an equation for the tangent to the curve at A. **4**

A4. Identify the locus in the complex plane given by $|z + i| = 2$. **3**

A5. Use the substitution $x = 1 + \sin\theta$ to evaluate $\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)^3}\,d\theta$. **5**

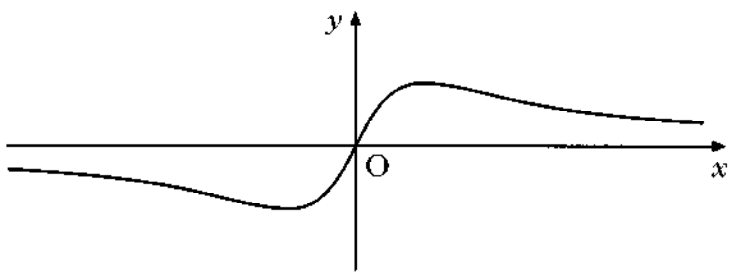
A6. Use elementary row operations to reduce the following system of equations to upper triangular form

$$\begin{aligned} x + y + 3z &= 1 \\ 3x + ay + z &= 1 \\ x + y + z &= -1. \end{aligned}$$ **2**

Hence express x, y and z in terms of the parameter a. **2**

Explain what happens when $a = 3$. **2**

A7.



The diagram shows the shape of the graph of $y = \frac{x}{1+x^2}$. Obtain the stationary points of the graph. **4**

Sketch the graph of $y = \left|\frac{x}{1+x^2}\right|$ and identify its three critical points. **3**

Marks

A8. Given that $p(n) = n^2 + n$, where n is a positive integer, consider the statements:

A $p(n)$ is always even
B $p(n)$ is always a multiple of 3.

For each statement, prove it if it is true or, otherwise, disprove it. **4**

A9. Given that $w = \cos\theta + i\sin\theta$, show that $\frac{1}{w} = \cos\theta - i\sin\theta$. **1**

Use de Moivre's theorem to prove $w^k + w^{-k} = 2\cos k\theta$, where k is a natural number. **3**

Expand $(w + w^{-1})^4$ by the binomial theorem and hence show that

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$ **5**

A10. Define $I_n = \int_0^1 x^n e^{-x}dx$ for $n \geq 1$.

(*a*) Use integration by parts to obtain the value of $I_1 = \int_0^1 xe^{-x}dx$. **3**

(*b*) Similarly, show that $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$. **4**

(*c*) Evaluate I_3. **3**

A11. The volume $V(t)$ of a cell at time t changes according to the law

$$\frac{dV}{dt} = V(10 - V) \qquad \text{for } 0 < V < 10.$$

Show that

$$\frac{1}{10}\ln V - \frac{1}{10}\ln(10 - V) = t + C$$

for some constant C. **4**

Given that $V(0) = 5$, show that

$$V(t) = \frac{10e^{10t}}{1 + e^{10t}}.$$ **3**

Obtain the limiting value of $V(t)$ as $t \to \infty$. **2**

[*END OF SECTION A*]

Candidates should now attempt ONE of the following

Section B (Mathematics 3) on *Page four*
Section C (Statistics 1) on *Pages five* and *six*
Section D (Numerical Analysis 1) on *Pages seven* and *eight*
Section E (Mechanics 1) on *Pages nine, ten* and *eleven.*

Section B (Mathematics 3)

Marks

ONLY candidates doing the course Mathematics 1, 2 and 3 should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

B1. Find the point of intersection of the line

$$\frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+1}{2}$$

and the plane with equation $2x + y - z = 4$. **4**

B2. The matrix A is such that $A^2 = 4A - 3I$ where I is the corresponding identity matrix. Find integers p and q such that

$$A^4 = pA + qI.$$ **4**

B3. A recurrence relation is defined by the formula

$$x_{n+1} = \frac{1}{2}\left\{x_n + \frac{7}{x_n}\right\}.$$

Find the fixed points of this recurrence relation. **3**

B4. Obtain the Maclaurin series for $f(x) = \sin^2 x$ up to the term in x^4. **4**
Hence write down a series for $\cos^2 x$ up to the term in x^4. **1**

B5. (*a*) Prove by induction that for all natural numbers $n \geq 1$

$$\sum_{r=1}^{n} 3(r^2 - r) = (n-1)n(n+1).$$ **4**

(*b*) Hence evaluate $\sum_{r=11}^{40} 3(r^2 - r)$. **2**

B6. Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x,$$

given that $y = 2$ and $\frac{dy}{dx} = 1$, when $x = 0$. **10**

[*END OF SECTION B*]

Section C (Statistics 1)

Marks

ONLY candidates doing the course Mathematics 1, 2 and Statistics 1 should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

C1. A mammogram is used to screen women for breast cancer. A mammogram which indicates an abnormality in the breast tissue is termed positive. Over many years, it has been determined that

(i) of all women screened, 1% have breast cancer,

(ii) P(Mammogram is positive | woman has breast cancer) = 0·9, and

(iii) P(Mammogram is negative | woman does not have breast cancer) = 0·9.

If a woman is screened and the mammogram is positive, find the probability that she actually has the disease. **5**

C2. A building society manager discovered that documentation for the proportion p = 0·25 of mortgage agreements required amendments by senior staff before final processing. Following a training workshop for staff involved in creating the documentation, the manager took a random sample of 20 completed agreements and found that in 3 cases amendments had been required.

(*a*) On the assumption that the training was ineffective, state the distribution and its parameters of the number of agreements requiring amendment, X, in random samples of 20. **2**

(*b*) Obtain $P(X \leq 3)$. **1**

The manager believed that the training had been effective since only 15% of the sample of agreements following the training had required amendment.

(*c*) Test the hypothesis p = 0·25, against the alternative p < 0·25. Indicate whether or not your conclusion supports the manager's belief. **3**

C3. A biologist found a report which stated that the body temperature for a species of mammal was normally distributed with mean 104 °F and standard deviation of 1·2 °F. He wished to convert this information to degrees Celsius.

(*a*) Given that x °F is equivalent to y °C, where $y = \frac{5}{9}(x - 32)$, obtain the exact values of the mean and standard deviation of the mammal's body temperature in °C. **4**

(*b*) Calculate the normal range for this animal's body temperature in °C, ie the range of temperatures symmetrically placed around the mean which includes 95% of body temperatures. **2**

[Turn over

Marks

C4. (*a*) Write down an expression for an approximate 95% confidence interval for a population proportion p. **2**

(*b*) The proportion of smokers, p, in a population is known to be of the order of 0·3.

(i) Show that the width of a 95% confidence interval for p, constructed from a sample of size n, will be of the order of $\frac{1 \cdot 8}{\sqrt{n}}$. **2**

(ii) Find the value of n required to estimate the true proportion to within ±0·05 with 95% confidence. **2**

C5. Bottles have burst strengths which are distributed with mean 502 psi and standard deviation 63 psi. A sample of 25 bottles with a new glass formulation was found to have a mean burst strength of 530 psi.

(*a*) Use a z-test with an appropriate critical region to investigate, at the 1% level of significance, whether or not the data provide evidence that mean burst strength has increased. **5**

(*b*) Calculate the p-value of the test and indicate how it can be used to confirm your decision in part (*a*). **2**

(*c*) Given that burst strength distributions are typically highly skewed, explain whether or not this would lead you to modify your earlier conclusion. **2**

[*END OF SECTION C*]

Section D (Numerical Analysis 1)

Marks

ONLY candidates doing the course Mathematics 1, 2 and Numerical Analysis 1 should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

D1. The following data are available for a function f:

x	1	4	6
$f(x)$	3·2182	4·0631	3·1278

Use the Lagrange interpolation formula to estimate $f(2{\cdot}5)$. **3**

D2. The function f is defined for $x > -1{\cdot}5$ by $f(x) = \ln(3 + 2x)$.

The polynomial p is the Taylor polynomial of degree two for the function f near $x = 1$. Express $p(1 + h)$ in the form $c_0 + c_1h + c_2h^2$. **3**

Use this polynomial to estimate the value of $\ln(5{\cdot}4)$ to four decimal places. **2**

State, with a reason, whether or not $f(x)$ is sensitive to small changes in x in the neighbourhood of $x = 1$. **1**

D3. In the usual notation for forward differences of function values $f(x)$ tabulated at equally spaced values of x,

$$\Delta f_i = f_{i+1} - f_i,$$

where $f_i = f(x_i)$ and $i = \ldots -2,\ -1,\ 0,\ 1,\ 2,\ \ldots$

Show that $\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0$. **2**

If each value of f_i is subject to an error whose magnitude is less than or equal to ε, determine the magnitude of the maximum possible rounding error in $\Delta^3 f_0$. **1**

When would this maximum possible error occur? **1**

D4. The following data (accurate to the degree implied) are available for a function f:

x	0·3	0·6	0·9	1·2	1·5	1·8
$f(x)$	1·298	1·195	1·323	1·700	2·346	3·280

(*a*) Construct a difference table of third order for the data. **3**

(*b*) Taking $x_0 = 0{\cdot}3$, identify the value $\Delta^2 f_3$. **1**

(*c*) State the degree of the polynomial which would best approximate this function. **1**

(*d*) Using the Newton forward difference formula of degree three, and working to three decimal places, obtain an approximation to $f(0{\cdot}63)$. **3**

Marks

D5. (*a*) Using a Taylor polynomial of degree two, or otherwise, derive the trapezium rule over a single strip and the corresponding principal error term. **5**

(*b*) Use the composite trapezium rule with four strips to obtain an estimate for the integral

$$\int_{\pi/4}^{\pi/2} x\sin x \, dx.$$

Perform the calculations using four decimal places. **3**

(*c*) Given that for $f(x) = x\sin x$, $f''(x) = 2\cos x - x\sin x$, and that $f'''(x)$ has no zero on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, obtain an estimate of the maximum truncation error in the integral. **2**

Hence state the value of the integral to a suitable accuracy. **1**

[*END OF SECTION D*]

Section E (Mechanics 1)

Marks

ONLY candidates doing the course Mathematics 1, 2 and Mechanics 1 should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

Where appropriate, candidates should take the magnitude of the acceleration due to gravity as $9{\cdot}8\,\text{m s}^{-2}$.

E1. (*a*) A particle moves on a straight line from the origin with initial velocity $U\mathbf{i}\ \text{m s}^{-1}$ and uniform acceleration $a\mathbf{i}\ \text{m s}^{-2}$, where $\mathbf{i}$ is the unit vector in the direction of motion.

Show, using calculus, that the distance $s(t)$ metres travelled by the particle in time t seconds is given by

$$s(t) = Ut + \tfrac{1}{2}at^2,$$

where t is measured from the start of the motion. **2**

(*b*) A ball is dropped from the top of a building of height H metres. The ball falls vertically from rest to the ground in 6 seconds.

Ignoring the effect of air resistance, calculate the time taken for the ball to reach a point halfway down the building. **2**

E2. An aircraft travels at 210 km/h in still air. The aircraft takes off from airfield A and lands at airfield B, where B is on a bearing of 050° from A.

Find the course the pilot must set in order to reach B if there is a steady wind blowing from the west at 30 km/h. **4**

E3. A car of mass m kg is travelling along a straight road at a constant velocity of $12\mathbf{i}\ \text{m s}^{-1}$, where $\mathbf{i}$ is the unit vector in the direction of motion. The driver of the car applies the brakes which produce a retarding force $-2m\left(1+\frac{t}{4}\right)\mathbf{i}$ newtons, where t is the time measured in seconds from the moment that the brakes are applied. The brakes are applied until the car is stationary.

Determine:

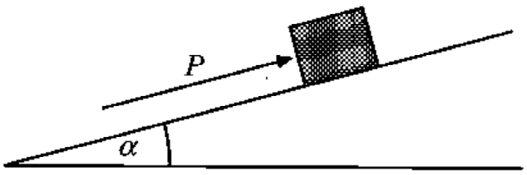
(*a*) the time taken for the car to stop; **5**

(*b*) the stopping distance. **2**

[Turn over

Marks

E4. A block of wood of mass m kg is at rest on a plane inclined at α to the horizontal as shown below, where $\tan\alpha = \frac{3}{4}$. A force of magnitude P newtons acting on the block parallel to the inclined plane, up the line of greatest slope, is just sufficient to prevent the block from sliding **down** the plane. The coefficient of friction between the block and the plane is μ.



(*a*) Show that

$$P = \frac{mg}{5}(3 - 4\mu),$$

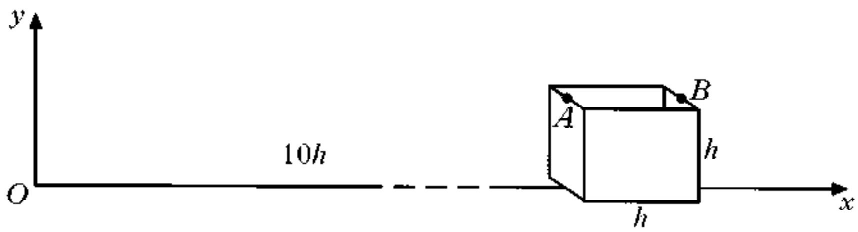
where g m s^{-2} is the magnitude of the acceleration due to gravity. **3**

(*b*) The force acting on the block parallel to the inclined plane is increased to $2P$ newtons and the block is now on the point of moving **up** the plane. Show that

$$P = \frac{mg}{10}(3 + 4\mu),$$

and hence find the value of μ. **4**

E5. A competition is held at a school gala. The object is to hit a golf ball from a point O on a horizontal playing field directly into an open box situated at a distance $10h$ metres away. The box is a cube with edges h metres long. A and B are the midpoints of the upper edges of the box as shown in the diagram in which AB is in the same plane as the x and y axes.



One of the pupils, Joanna, hits the ball from O, in the vertical plane OAB, imparting a speed of V m s^{-1} to the ball with angle of projection 45°.

(*a*) Using the coordinate system shown in the diagram, show that the equation of the trajectory of the ball is

$$y = x - \frac{gx^2}{V^2},$$

where g m s^{-2} is the magnitude of the acceleration due to gravity. **4**

Marks

E5. (continued)

(*b*) Obtain an expression for V, in terms of g and h, for the ball to hit A. **3**

(*c*) Suppose that Joanna succeeds in hitting the ball into the box. Show that the speed of projection satisfies

$$\frac{10}{3} < \frac{V}{\sqrt{gh}} < \frac{11}{\sqrt{10}}.$$ **3**

[*END OF SECTION E*]

[*END OF QUESTION PAPER*]