

X100/701

NATIONAL
QUALIFICATIONS
2006

FRIDAY, 19 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**



Answer all the questions.

1. Calculate the inverse of the matrix $\begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}$.
For what value of x is this matrix singular? 4
2. Differentiate, simplifying your answers:
(a) $2 \tan^{-1} \sqrt{1+x}$, where $x > -1$; 3
(b) $\frac{1+\ln x}{3x}$, where $x > 0$. 3
3. Express the complex number $z = -i + \frac{1}{1-i}$ in the form $z = x + iy$, stating the values of x and y . 3
Find the modulus and argument of z and plot z and \bar{z} on an Argand diagram. 4
4. Given $xy - x = 4$, use implicit differentiation to obtain $\frac{dy}{dx}$ in terms of x and y . 2
Hence obtain $\frac{d^2y}{dx^2}$ in terms of x and y . 3
5. Obtain algebraically the fixed point of the iterative scheme given by
$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{2}{x_n^2} \right), \quad n = 0, 1, 2, \dots$$
 3
6. Find $\int \frac{12x^3 - 6x}{x^4 - x^2 + 1} dx$. 3
7. For all natural numbers n , prove whether the following results are true or false.
(a) $n^3 - n$ is always divisible by 6.
(b) $n^3 + n + 5$ is always prime. 5
8. Solve the differential equation
$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0,$$

given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 2$. 6

9. Use Gaussian elimination to obtain solutions of the equations

$$\begin{aligned} 2x - y + 2z &= 1 \\ x + y - 2z &= 2 \\ x - 2y + 4z &= -1. \end{aligned}$$

5

10. The amount x micrograms of an impurity removed per kg of a substance by a chemical process depends on the temperature $T^\circ\text{C}$ as follows:

$$x = T^3 - 90T^2 + 2400T, \quad 10 \leq T \leq 60.$$

At what temperature in the given range should the process be carried out to remove as much impurity per kg as possible?

4

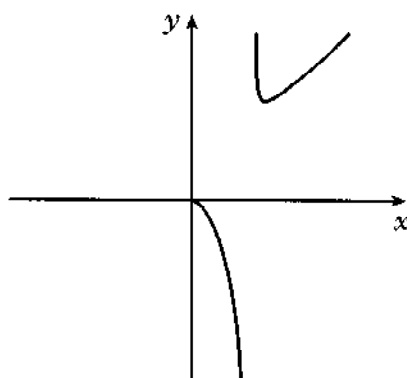
11. Show that $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$, where $0 < \theta < \frac{\pi}{2}$.

1

By expressing $y = \cot^{-1} x$ as $x = \cot y$, obtain $\frac{dy}{dx}$ in terms of x .

3

- 12.



The diagram shows part of the graph of a function f which satisfies the following conditions:

- (i) f is an even function;
- (ii) two of the asymptotes of the graph $y = f(x)$ are $y = x$ and $x = 1$.

Copy the diagram and complete the graph. Write down equations for the other two asymptotes.

3

13. The square matrices A and B are such that $AB = BA$. Prove by induction that $A^n B = B A^n$ for all integers $n \geq 1$.

5

[Turn over for Questions 14 to 17 on Page four]

14. (a) Determine whether $f(x) = x^2 \sin x$ is odd, even or neither. Justify your answer. 3
- (b) Use integration by parts to find $\int x^2 \sin x \, dx$. 4
- (c) Hence find the area bounded by $y = x^2 \sin x$, the lines $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$ and the x -axis. 3

15. Obtain an equation for the plane passing through the point $P(1, 1, 0)$ which is perpendicular to the line L given by

$$\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1}. \quad 3$$

Find the coordinates of the point Q where the plane and L intersect. 4

Hence, or otherwise, obtain the shortest distance from P to L and explain why this is the shortest distance. 2, 1

16. The first three terms of a geometric sequence are

$$\frac{x(x+1)}{(x-2)}, \quad \frac{x(x+1)^2}{(x-2)^2} \quad \text{and} \quad \frac{x(x+1)^3}{(x-2)^3}, \quad \text{where } x < 2.$$

- (a) Obtain expressions for the common ratio and the n th term of the sequence. 3
- (b) Find an expression for the sum of the first n terms of the sequence. 3
- (c) Obtain the range of values of x for which the sequence has a sum to infinity and find an expression for the sum to infinity. 4
17. (a) Show that $\int \sin^2 x \cos^2 x \, dx = \int \cos^2 x \, dx - \int \cos^4 x \, dx$. 1

- (b) By writing $\cos^4 x = \cos x \cos^3 x$ and using integration by parts, show that

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{1}{4} + 3 \int_0^{\pi/4} \sin^2 x \cos^2 x \, dx. \quad 3$$

- (c) Show that $\int_0^{\pi/4} \cos^2 x \, dx = \frac{\pi+2}{8}$. 3

- (d) Hence, using the above results, show that

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{3\pi+8}{32}. \quad 3$$

[END OF QUESTION PAPER]